Recent Advances on Robust Tensor Principal Component Analysis with T-SVD

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Abstract

The task of robust tensor principal component analysis (RTPCA) is to separate the underlying low-rank component and sparse component in high-dimensional data. In RTPCA, an order-3 tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ can be decomposed as $\mathcal{X} = \mathcal{L} + \mathcal{E}$, where $\mathcal{L}$ and $\mathcal{E}$ represent a low-rank tensor and a sparse tensor, respectively. Compared with traditional RPCA methods for two-way data, RTPCA can make good use of the multi-dimensional structure, has found successful applications in background modeling, image denoising, illumination normalization for face images, etc.

The key of RTPCA problem is how to recover $\mathcal{L}$ and $\mathcal{S}$ accurately. Several different sparse constraints are adopted according to different applications [Zhang *et al.*, 2014; Zhou and Feng, 2017]. Among them, the most commonly used one is the ℓ_1 norm. Meanwhile, a number of tensor decompositions produce diverse tensor ranks for low-rank component estimation. Unlike the traditional canonical polyadic (CP) decomposition and Tucker decomposition, the recently proposed tensor singular value decomposition (t-SVD) has lower computational complexity by employing fast Fourier tranform in the 3rd mode. As the corresponding tubal rank minimization problem is highly non-convex, the rank term is usually relaxed into a convex tensor nuclear norm (TNN). The RTPCA based on standard t-SVD can be formulated as follows [Lu *et al.*, 2016; Lu *et al.*, 2019]:

$$\underset{\mathcal{L},\, \mathcal{E}}{\text{minimize}}\ \|\mathcal{L}\|_{\text{TNN}} + \lambda \|\mathcal{E}\|_1, \text{ s. t. } \mathcal{X} = \mathcal{L} + \mathcal{E} \quad (1)$$

where λ makes balance of the low rank component and the sparse component. This optimization model can be solved by the alternating direction method of multipliers (ADMM), which is transformed into two subproblems including low-rank approximation and sparse approximation.

This extended abstract gives a brief survey of the recent advances on RTPCA. In classical ways, the observed tensor is processed directly in its original scale. However, the original scale may not be the optimal one for analysis. In [Chen *et al.*, 2017; Feng *et al.*, 2020], the whole tensor is split into a number of small block tensors and low rank component is extracted in blocks. The robust block tensor principal component analysis (RBTPCA) can be formulated into:

$$\begin{aligned} &\min_{\mathcal{L}_p, \mathcal{E}} \sum_{p=1}^{P} \|\mathcal{L}_p\|_{\text{TNN}} + \lambda \|\mathcal{E}\|_1 \\ &\text{s. t. } \mathcal{X} = \mathcal{L}_1 \boxplus \mathcal{L}_2 \boxplus \cdots \boxplus \mathcal{L}_P + \mathcal{E} \end{aligned} \quad (2)$$

where P is the number of small blocks divided from the original data; $\boxplus$ represents the concatenation operator of blocks; $\mathcal{L}_p$ is the low rank component of block tensor $\mathcal{X}_p$, $p = 1, 2, \cdots, P$; $\mathcal{E}$ denotes the sparse component. RBTPCA could utilize the similarity of local pixels to extract more details. The iterative block tensor singular value threshold operator has been strictly derived for the low-rank approximation subproblem and the block tensor incoherence conditions have been given to guarantee the successful recovery. In addition, we analyze the effect of block size on accuracy and convergence speed. Experimental results show a carefully selected scale can improve the performance. Especially on illumination normalization for face images, RBTPCA can remove almost all shadows on face, which is meaningful for face detection.

On the other hand, t-SVD has some drawbacks and the classic TNN could not extract the low rank component very well. The t-SVD of $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ can be represented as $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^{\text{T}}$, and its computation process can be summarized as follows:

1. Taking the Discrete Fourier Transformation (DFT) of $\mathcal{A}$ along the third mode to obtain $\hat{\mathcal{A}}$;
2. Computing matrix SVD of each frontal slice of $\hat{\mathcal{A}}$ in the first and second dimensions;
3. Taking the Inverse Discrete Fourier Transformation (IDFT).

As the t-SVD deals with the first and second modes of tensor data through the matrix SVD but leaves the third mode by DFT, the low rank information on third mode may not be fully exploited. At

least the performance may vary due to the rotation variance of the t-SVD. [Chen *et al.*, 2018; Liu *et al.*, 2018] find low rank structure still exists in the core tensor $\mathcal{S}$. To further exploit the low rank structure, the low rank component is further extracted from the core matrix $\mathbf{S}$ whose entries are from the diagonal elements of the core tensor $\mathcal{S}$. Defining $\mathcal{S} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ and $I = \min(I_1, I_2)$, the core matrix $\mathbf{S} \in \mathbb{R}^{I \times I_3}$ satisfies $\mathbf{S}(i,:) = \mathcal{S}(i,i:)$. Specifically, combining the matrix nuclear norm of core matrix with the classical TNN, the low rank structure in all modes of the target tensor will be better characterized. The obtained improved tensor nuclear norm (ITNN) is defined as follows:

$$\|\mathcal{A}\|_{\mathrm{ITNN}} = \|\mathcal{A}\|_{\mathrm{TNN}} + \lambda_S \|\mathbf{S}\|_* \tag{3}$$

where λ_S is a parameter to balance the two terms. In contrast to the classical TNN, the additional term $\|\mathbf{S}\|_*$ can additionally exploit low rank information in the third mode. The proposed ITNN tries to take advantage of structural feature of tensor data as completely as possible.

Based on the newly defined ITNN, the improved robust tensor principal component analysis (IRTPCA) optimization model is formulated as:

$$\begin{aligned} &\min_{\mathcal{L},\,\mathcal{E}} \|\mathcal{L}\|_{\mathrm{ITNN}} + \lambda \|\mathcal{E}\|_1 \\ &\text{s. t. } \ \mathcal{X} = \mathcal{L} + \mathcal{E} \end{aligned} \tag{4}$$

By additionally exploiting correlation in the third mode, the IRTPCA achieves better performance in a series of image processing applications. Especially in background extraction, because the low rank structure of video mainly lies in the third mode due to the correlation between frames, the additional low rank extraction can effectively outperform the classical one with standard TNN.

As introduced before, t-SVD can be computed efficiently by DFT and matrix SVD. Different frontal slices of data in the Fourier domain have vary frequency characteristics. For example, a color image has three color channels and there will be two frequency bands in Fourier domain. One is called zero-frequency band, which always contains the most texture information of this image. The other always indicates the difference information of three channels. In order to better utilize the prior knowledge of frequency spectrum in the t-SVD based low-rank component estimation, [Wang *et al.*, 2020] propose frequency-weighted tensor nuclear norm (FTNN) by performing frequency component analysis on the 3rd mode. The frequency-weighting vector depends on the prior knowledge of data in the Fourier domain.

The classical TNN based RTPCA can be regarded as a special case of these three generalized ones, i. e. RBTPCA, IRTPCA, FTNN-RTPCA. In the future, multi-scale concept can be integrated into more versions of RTPCA. The low rank approximation based on ITNN can be generalized into higher-order tensor data to deal with the other problem of t-SVD that it can only process 3-order tensor. The frequency analysis on the 3rd can be modified into data-driven version. The omni-directional total variation can replace the classical sparse constraints too [Liu *et al.*, 2017].

Besides of t-SVD and its variants, some other decompositions are used in RTPCA, such as tensor train, tensor tree. They may achieve better accuracy in low rank tensor component estimation, but the computational efficiency is inferior. Those methods can be applied when the accuracy is much more important than computational efficiency.

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